# On the Bachelier implied volatility at extreme strikes

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## Lognormal process

A drift-less underlying asset price $F$ is subject to the following stochastic differential equation:

$$\mathrm{d}F = \sigma F \,\mathrm{d}W \,,$$

## Black76 formula

The value at $t$ of a European option with payoff $V(T) = \eta n \max(S(T) - K, 0)$ reads (Black, 1976):

$$V(t) = \eta n B(t, T_d) \left[ F(t, T) \Phi\left(\eta d_1\left(F(t, T), K\right)\right) - K \Phi\left(\eta d_2\left(F(t, T), K\right)\right) \right]$$

where

$$d_1(F, K) = \frac{\ln \frac{F}{K} + \frac{1}{2}\bar{\sigma}^2 (T - t)}{\bar{\sigma}\sqrt{T - t}}\,, \quad d_2 = d_1 - \bar{\sigma}\sqrt{T - t},$$

- $\bar{\sigma}$ may include stochastic interest rate effects.
- additional discount term for the premium payment date.

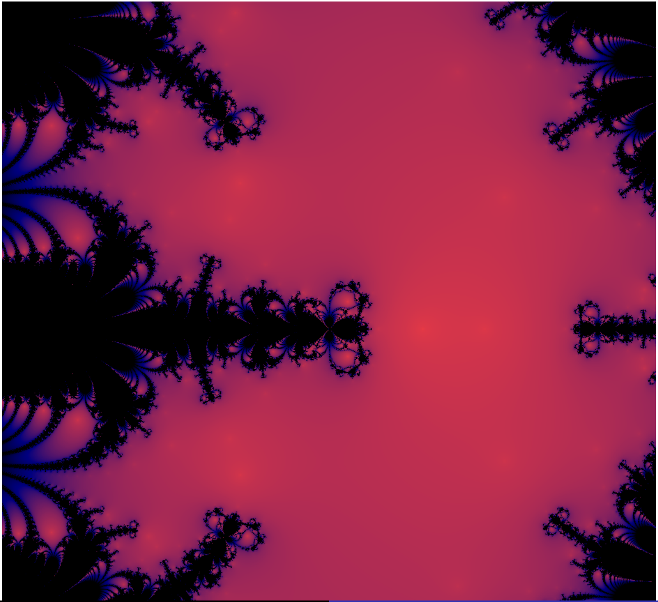
Numerical solver: Newton, Halley, Householder (1972)
Good initial guess: Stefanica and Radoicic (2017) based on Polya approximation (1949)

$$\Phi(x) \approx \frac{1}{2} + \frac{1}{2} sgn(x) \sqrt{1 - e^{-\frac{2}{\pi}x^2}},$$

leads to an explicit initial guess.
Described in details in Healy's (2021) book

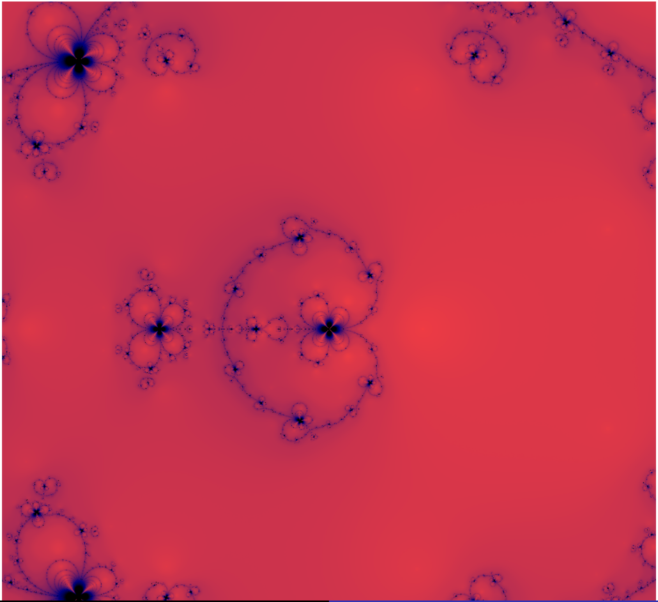
# Black-Scholes

## Implied volatility fractals - Newton



# Black-Scholes

## Implied volatility fractals - Householder



- Markets quote prices. Some brokers/market data providers give IV directly.
- Interpolate and manage (OTC) options with different strikes/maturities. Or binary options and more exotic options.
- stochastic vol models calibration: a better scale than price.
- local vol model.

The Dupire local volatility formula (Dupire 1994):

$$\sigma^{\star 2}(K,T) = \frac{1}{2}\frac{\frac{\partial C_0}{\partial T}}{K^2\frac{\partial^2 C_0}{\partial K^2}}. \quad (1)$$

The local volatility $\sigma^{\star 2}(y,T)$ in terms of the total variance $w(y,T)=\bar{\sigma}^2 T$ (Gatheral, 2006)

$$\sigma^{\star 2}(y,T) = \frac{\frac{\partial w}{\partial T}}{1-\frac{y}{w}\frac{\partial w}{\partial y}+\frac{1}{4}\left(-\frac{1}{4}-\frac{1}{w}+\frac{y^2}{w^2}\right)\left(\frac{\partial w}{\partial y}\right)^2+\frac{1}{2}\frac{\partial^2 w}{\partial y^2}}. \quad (2)$$

Hodges (1996), for $K > F$ (resp. $K < F$)

$$\sigma(K) = \sqrt{2/T\ln(K/F)}, \sigma(K) = \sqrt{-2/T\ln(K/F)}$$

Lee (2002), Benaim, Fritz and Lee (2008) "The moment formula"

$$\tilde{p} = \sup\left\{p : \mathbb{E}\left[F(T)^{p+1}\right] < \infty\right\}, \quad \beta_R = \limsup_{x\to\infty} \frac{\sigma^2(x)}{|x|/T}$$

where $x = \ln\frac{K}{F}$.
Then $\beta_R \in [0, 2]$ and $\tilde{p} = \frac{1}{2\beta_R} + \frac{\beta_R}{8} - \frac{1}{2}$.

## Normal process

A drift-less underlying asset price $F$ is subject to the following stochastic differential equation:

$$\mathrm{d}F = \sigma_N \,\mathrm{d}W \,,$$

## Bachelier formula (1900) for a call option

$$V(K,\sigma_N) = B(0,T)\left[(F-K)\Phi\left(\frac{F-K}{\sigma_N\sqrt{T}}\right) + \sigma_N\sqrt{T}\phi\left(\frac{F-K}{\sigma_N\sqrt{T}}\right)\right],$$

- CHF rates became negative in 2009
- EUR rates negative in 2014-2021

REPORT DATE

01-Dec-2014

CURRENCY

EUR

SUBMIT

ICE Libor Historical Rates

| TENOR | EUR ICE LIBOR 01-DEC-2014 |
|---|---|
| Overnight | -0.02857 |
| 1 Week | -0.01714 |
| 1 Month | 0.01143 |

- Negative commodity prices April 2020.

- Negative commodity prices April 2020.

  > *Crude oil futures prices for May 2020 delivery were traded below zero on April 20 and 21, reaching a minimum of −40.32 USD/barrel.*

- Same concept as in Black-Scholes.
- Nicer inversion: $x/C(x)$ can be reduced to one variable using $d = x/\sigma\sqrt{T}$ with $x = F - K$. Chebyshev or rational approximation to machine epsilon available (Le Floc'h 2016).

## Asymptotic upper bound

The Bachelier implied volatility $\sigma_N$ is bounded above by $\frac{K-F}{\sqrt{2T\ln K}}$ when $K \to +\infty$ or more precisely, if $\exists b < 2 \mid \forall K \in \mathbb{R}^+, \exists K_0 > K$ such that $\sigma_N(K_0, T) > \frac{K_0-F}{\sqrt{bT\ln K_0}}$ then the Bachelier option price $C_N(K, T, \sigma_N)$ has arbitrages.

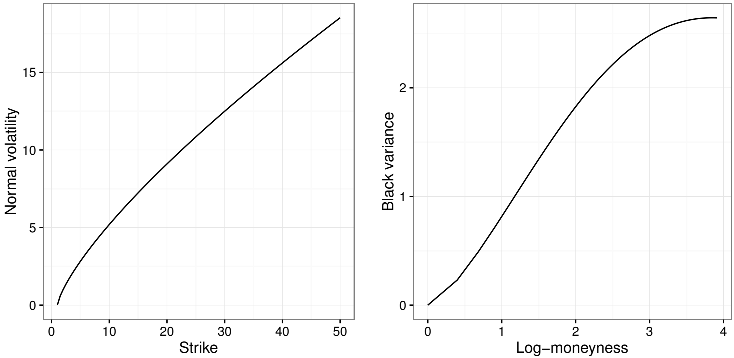
## Acceptable lower bound

If $\sigma_N(K, T) = \frac{K-F}{\sqrt{bT\ln(K-F)}}$ for $b \geq 2$, then $\exists K_0 \mid \forall K > K_0, C_N(K, T, \sigma_N(K))$ has no arbitrages.

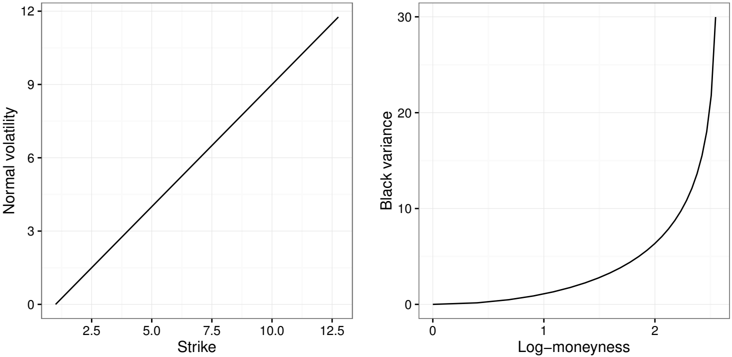
## Relation between moments explosion and the Bachelier implied volatility

If the Bachelier implied volatility is below $v_R(K) = \frac{K-F}{\sqrt{bT \ln K}}$ as $K \to \infty$ with $b > 2(1+p)$ and below $v_L(K) = \frac{|K|+F}{\sqrt{cT \ln |K|}}$ as $K \to -\infty$ with $c > 2(1+p)$, then the $(p+1)$-th moment $\mathbb{E}\left[F^{p+1}\right]$ exists.

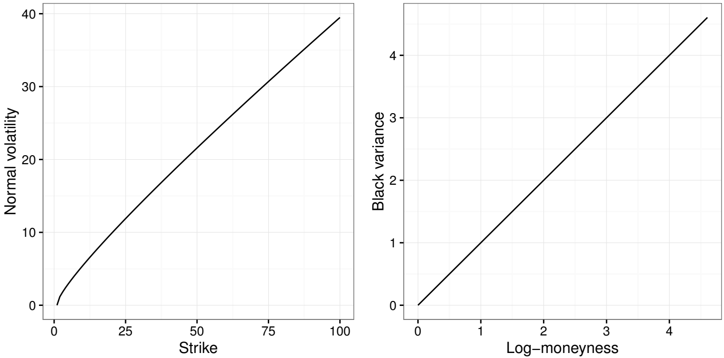
proof: based on Carr-Madan replication formula.




Bachelier left, Black right $\sigma_N = (K - f)^{\frac{3}{4}}$.


12
9
6
3
0
Normal volatility
2.5
5.0
7.5
10.0
12.5
Strike
30
20
10
0
Black variance
0
1
2
Log−moneyness

$$\sigma_N = (K - f)$$




Bachelier normal volatility corresponding to the Black variance $w = \ln \frac{K}{F}$.

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