Finite difference techniques for arbitrage-free SABR

Fabien Le Floc'h, Gary Kennedy

Calypso Technology, Clarus Financial Technology

Lorentz Center - Conference on Models and Numerics in Financial Mathematics, 2015

Negative rates on the market
Examples

- CHF rates became negative in 2009
- EUR rates are negative now

REPORT DATE

01-Dec-2014

CURRENCY

EUR

SUBMIT

ICE Libor Historical Rates

TENOR	EUR ICE LIBOR 01-DEC-2014
Overnight	-0.02857
1 Week	-0.01714
1 Month	0.01143

The old way:

lognormal (Black) volatility

$$dF = F\sigma dW$$

New ways to quote caps and swaptions:

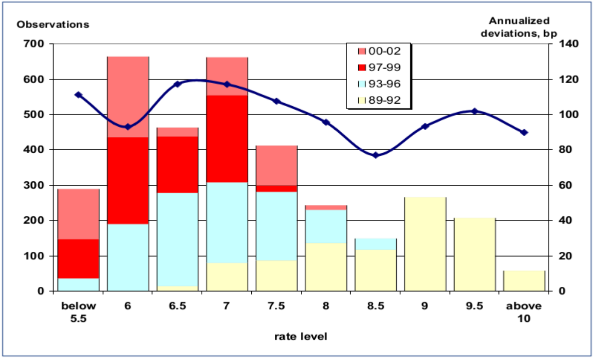
basis point volatility (b.p. vol)

$$dF = \sigma dW$$
$$\text{b.p. vol} = 100^2\sigma$$

shifted lognormal volatility

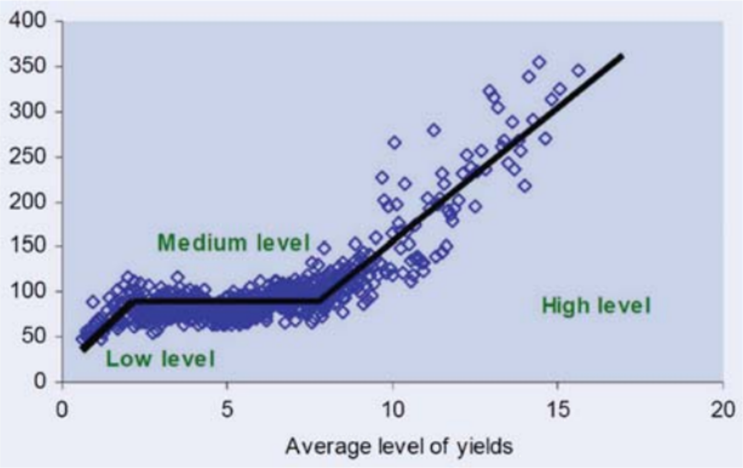
$$dF = (F + b)\sigma dW$$
$$\text{currently } b = 1\%.$$

Daily volatility vs 10y swap rates

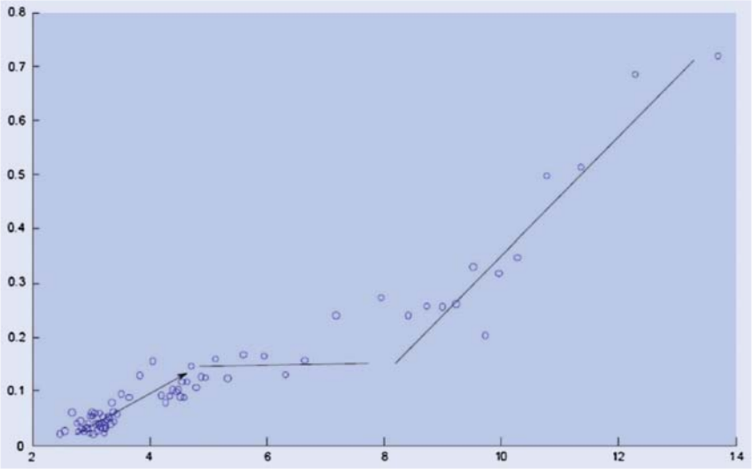


Levin, 2004 - "A weak or absent relation between absolute volatility and rate level is a sign of normality rather than lognormality. It also prompts quoting rate uncertainty (and, therefore, option prices) in terms of absolute volatility rather than relative volatility."

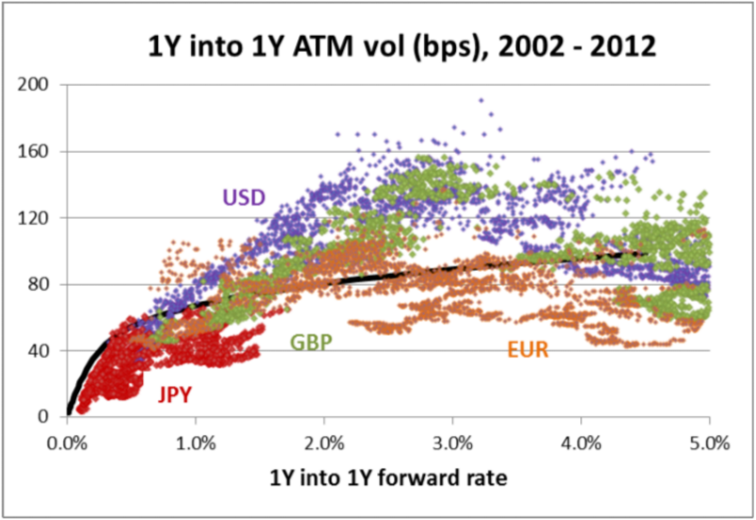
DeGuillaume, Rebonato, Pagudin (2013)
2y swap rates on USD, GBP, JPY, CHF for the last 40 years



DeGuillaume, Rebonato, Pagudin (2013)
UK Consol yield for the last 150 years



Hagan (2013)



1Y into 1Y ATM vol (bps), 2002 - 2012
200
160
120
80
40
0
0.0%
1.0%
2.0%
3.0%
4.0%
5.0%
1Y into 1Y forward rate
USD
GBP
EUR
JPY

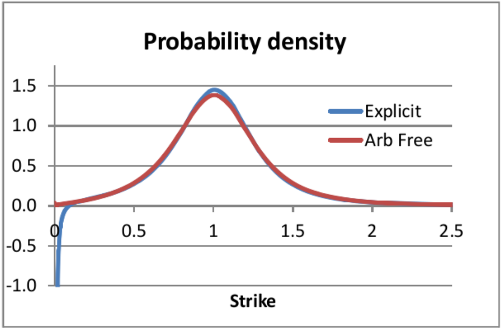


FIG. 2.3. *Probabiity density for the SABR model for* $\alpha = 35\%$, $\beta = 0.25$, $\rho = -10\%$, *and* $\nu = 100\%$. *Shown are the densities obtained from the explicit formulas for* $\sigma_N(K)$ *and from the arb free approach for* $\tau_{ex} = 1yr$.

- Happens regularly for medium and long term swaptions.
- Really a problem of the implied vol expansion, not of SABR.

Formula level fixes:

- Obloj (2008): fixes discontinuity when $\beta \to 1$ - still arbitrages.
- Benhamou Croissant (2007): a numerical integral - still arbitrages.
- Paulot (2009): 2nd order in time expansion - still arbitrages.
- Johnson Nonas (2009): blending to make wings lower for longer expiries - still arbitrages.

All those formulas are based on expansions, ignoring the absorption at 0, which is significant for low rates.

- A specific extrapolation
  - Benaim et al. $P(K) = K^{\mu} e^{a+bK+cK^2}$ fixes CMS convexity adjustment, CMS spread. But where to place μ and K?
  - could do the same with Grzelak stochastic collocation
- Numerical approaches
  - Andreasen Huge SABR/ZABR (2011): 1 step forward Dupire PDE - does not match classic SABR ATM
  - Doust (2012): density expansion. Absorption probability d_0 very involved numerically
  - Hagan (2013): PDE on the density.

Hagan 1D SABR density PDE

Fokker-Planck on the probability density Q

$$\frac{\partial Q}{\partial T}(T,F) = \frac{\partial^2 M(T,F)Q(T,F)}{\partial F^2} \text{ and } \begin{cases} \frac{\partial Q_L}{\partial T}(T) = \lim_{F\to F_{\min}} \frac{\partial MQ}{\partial F} \\ \frac{\partial Q_R}{\partial T}(T) = \lim_{F\to F_{\max}} \frac{\partial MQ}{\partial F} \end{cases}$$

$$M(T,F) = \frac{1}{2}D^2(F)E(T,F),\ E(T,F) = e^{\rho\nu\alpha\Gamma(F)T},\ \Gamma(F) = \frac{F^\beta - f^\beta}{F-f}$$

$$D(F) = \sqrt{\alpha^2 + 2\alpha\rho\nu y(F) + \nu^2 y(F)^2}F^\beta,\ y(F) = \frac{F^{1-\beta} - f^{1-\beta}}{1-\beta}$$

and initial condition

$$\lim_{T\to 0} Q(T,F) = \delta(F-f)$$

The maximum principle implies that the density stays positive

Dupire formulation

Corresponding normal forward Dupire PDE

$$\frac{\partial V_{call}}{\partial T}(T,F) = \frac{1}{2}D^2(F)E(T,F)\frac{\partial^2 V_{call}}{\partial F^2}(T,F)$$

with initial condition $V_{call}(0,F) = (f-F)^+$.

Very close to Andreasen Huge, but:

- In Andreasen Huge, $E(T,F) = 1$
- same order of expansion as the classic SABR formula
- with linear boundary condition $\frac{\partial^2 V_{call}}{\partial F^2}(F_{min}) = 0$
- multiple steps

Hagan's moment preserving finite difference scheme

Arbitrage free properties:

- No loss of probability at every time step
  $Q_L(t_i) + \int_{F_{min}}^{F_{max}} Q(t_i, u)du + Q_R(t_i) = 1$
- Martingale property preserved at every time step
  $F_{min}Q_L(t_i) + \int_{F_{min}}^{F_{max}} uQ(t_i, u)du + F_{max}Q_R(t_i) = f.$

but:

- Crank-Nicolson time marching:
  - known for oscillations on non smooth initial data
  - The maximum principle is only verified if the Courant number is low $\implies$ potentially high number of time steps. $\Psi_Q = M\frac{\delta}{h^2}$.
- Uniform grid
  - not always appropriate for long maturities
  - non uniform grid increases Courant number and likeliness of oscillations

Transformation of the Fokker-Planck PDE

$F_{max} > 13000$ for 3 std dev / $F_{max} > 300000$ for 4 std dev with $\alpha = 100\%, \beta = 0.30, \rho = 90\%, \nu = 100\%, \tau_{ex} = 10, f = 1$ (extreme)

The Lamperti transform - towards a unit diffusion

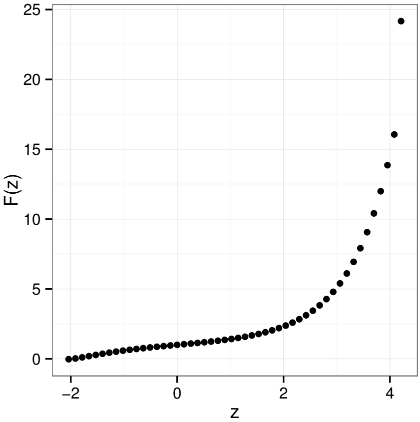
$$z(F) = \int_f^F \frac{dF'}{D(F')}$$

PDE in $\theta(z) = Q(F(z))B(z)$ with $B(z) = D(F(z))$

$$\frac{\partial \theta}{\partial T} = \frac{1}{2}\frac{\partial}{\partial z}\left\{\frac{1}{B}\frac{\partial BE\theta}{\partial z}\right\} \text{ and } \begin{cases}\theta(T,z) = 0 \text{ as } z \to z(F_{\min}) \\ \theta(T,z) = 0 \text{ as } z \to z(F_{\max})\end{cases}$$

$$y(z) = \frac{\alpha}{\nu}\left[\sinh(\nu z) + \rho(\cosh(\nu z) - 1)\right]$$

$$F(y) = \left[f^{1-\beta} + (1-\beta)y\right]^{\frac{1}{1-\beta}}$$

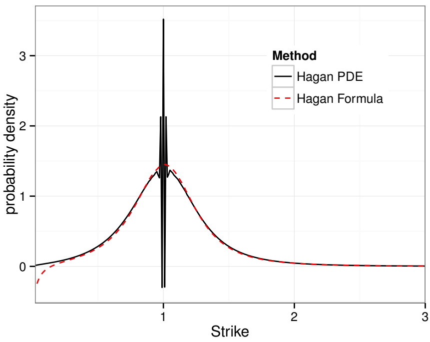


- Much higher accuracy for the same number of points.
- Still moment preserving.

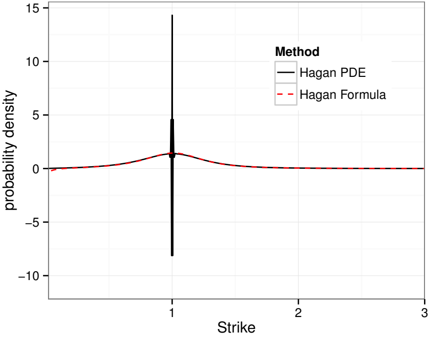
Crank-Nicolson oscillations

500 points. $\alpha = 35\%, \beta = 0.25, \rho = -10\%, \nu = 100\%, \tau_{ex} = 1$

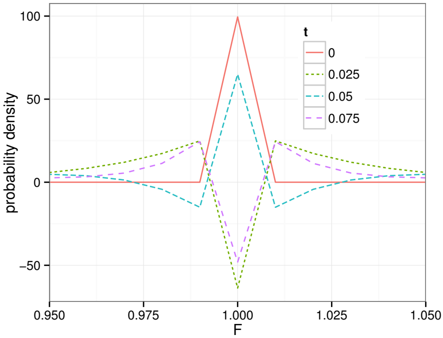
Q, 40 time steps and $F_{max} = 5$



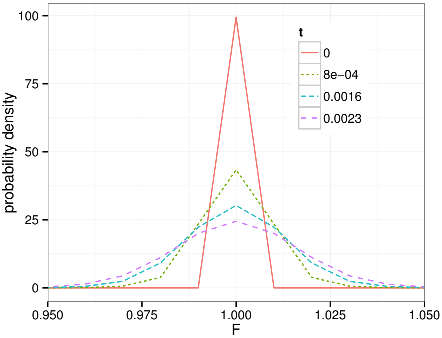
θ, 80 time steps and $n_{sd} = 4$



40 time steps $\Psi_Q = 15.16$



1280 time steps $\Psi_Q = 0.47$



4 half time steps of implicit Euler before Crank-Nicolson - Rannacher (1984), Giles and Carter (2005). Let $\mathcal{L}_j^n$ be the relevant discrete operator at time n and point j:

$$\theta_j^{n+\frac{1}{2}} - \theta_j^n = \frac{\delta}{2}\mathcal{L}_j^{n+\frac{1}{2}}\theta_j^{n+\frac{1}{2}}$$

$$P_L(t_{n+\frac{1}{2}}) - P_L(t_n) = \frac{\delta}{2}\frac{\hat{B}_1}{\hat{F}_1 - \hat{F}_0}\hat{E}_1(t_{n+\frac{1}{2}})\theta_1^{n+\frac{1}{2}}$$

$$P_R(t_{n+\frac{1}{2}}) - P_R(t_n) = \frac{\delta}{2}\frac{\hat{B}_J}{\hat{F}_{J+1} - \hat{F}_J}\hat{E}_J(t_{n+\frac{1}{2}})\theta_J^{n+\frac{1}{2}}$$

$$\theta_j^{n+1} - \theta_j^n = \frac{\delta}{2}\left(\mathcal{L}_j^{n+1}\theta_j^{n+1} + \mathcal{L}_j^n\theta_j^n\right)$$

- Not L-stable. How many smoothing steps are really needed?
- implicit Euler steps only order-1, and quite critical here.

On the simple problem:

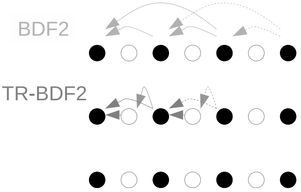
$$u'(t) = \lambda u(t), \lambda \in \mathbb{C} \tag{1}$$

Forward Euler: $u_{j+1} = (1 + k\lambda)u_j = (1 + z)u_j$
Backward Euler: $u_{j+1} = \frac{1}{1-k\lambda}u_j = \frac{1}{1-z}u_j$

- A-stable when the stability region $|\frac{u_{j+1}}{u_j}| \leq 1$ contains left half plane
- L-stable when $|\frac{u_{j+1}}{u_j}| \to 0$ when $|z| \to \infty$. Rapid transients in the solution will be damped in a single time step

BDF2

$$3\theta_j^{n+2} - 4\theta_j^{n+1} + \theta_j^n = 2\delta\mathcal{L}_j^{n+2}\theta_j^{n+2}$$



TR-BDF2

$$\theta_j^{n+\alpha} - \theta_j^n = \frac{\alpha\delta}{2}\left(\mathcal{L}_j^{n+\alpha}\theta_j^{n+\alpha} + \mathcal{L}_j^n\theta_j^n\right)$$

$$\theta_j^{n+1} = \frac{1}{2-\alpha}\left(\frac{1}{\alpha}\theta_j^{n+\alpha} - \frac{(1-\alpha)^2}{\alpha}\theta_j^n + \delta(1-\alpha)\mathcal{L}_j^{n+1}\theta_j^{n+1}\right)$$

Alternative Moment Preserving Schemes

L-stable schemes

Lawson-Swayne

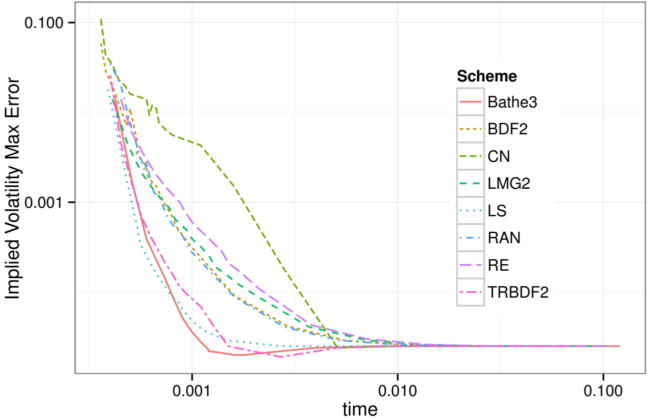
$$\theta_j^{n+b} - \theta_j^n = b\delta\mathcal{L}_j^{n+b}\theta_j^{n+b}$$

$$\theta_j^{n+2b} - \theta_j^{n+b} = b\delta\mathcal{L}_j^{n+2b}\theta_j^{n+2b}$$

$$\theta_j^{n+1} = (\sqrt{2}+1)\theta_j^{n+2b} - \sqrt{2}\theta_j^{n+b}$$

Lawson-Morris-Gourlay / Richardson

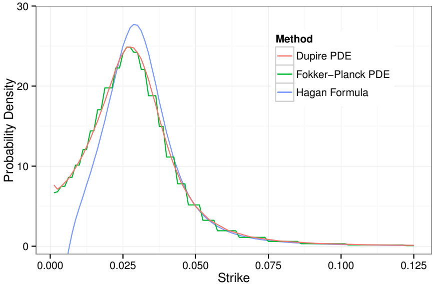
$$\theta(z) = 2\bar{\theta}^{\frac{\delta}{2}}(z) - \bar{\theta}^{\delta}(z)$$



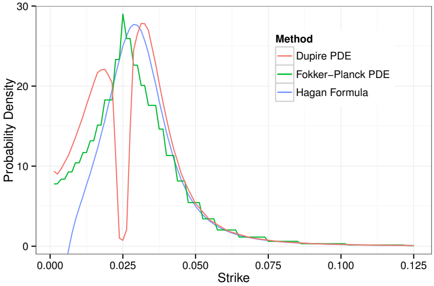
All better than CN. Lawson-Swayne gave best trade-off performance vs accuracy.

50 space steps

with 5 time-steps



with 2 time-steps

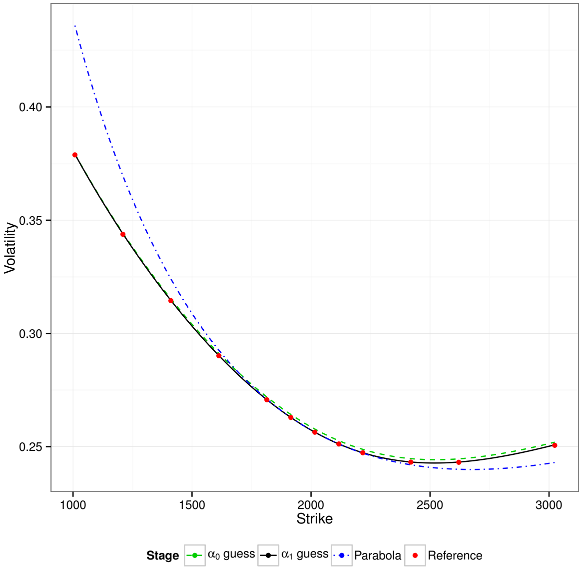


Taylor expansion of order-2 ATM (also in Hagan (2002))

$$\sigma_B(z) = \alpha(f+b)^{\beta-1} + \frac{1}{2}\left(\rho\nu - (1-\beta)\alpha(f+b)^{\beta-1}\right) z + \frac{1}{12\alpha(f+b)^{\beta-1}}\left((1-\beta)^2(\alpha(f+b)^{\beta-1})^2 + \nu^2(2-3\rho^2)\right) z^2$$

with $z = \log(K/f)$

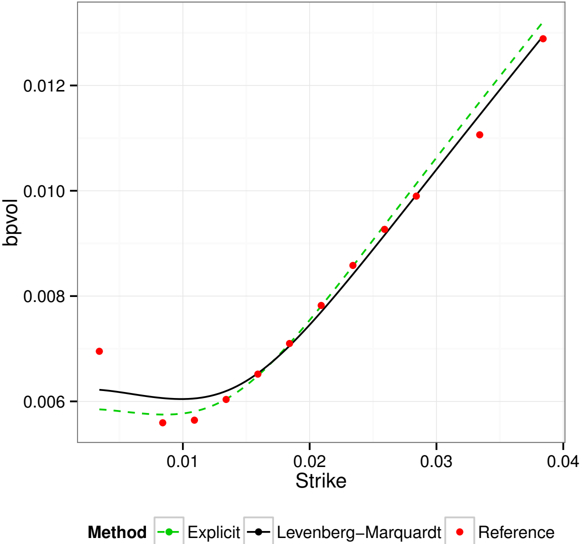
- Matching ATM vol, σ_0, ATM skew σ_0', ATM curvature σ_0'' leads to an analytically solvable 3 dimensional system in α, ρ, ν.
- Can be refined with West (2005) ATM cubic polynomial.
- Same approach with the normal formula.



0.40
0.35
0.30
0.25
Volatility
1000
1500
2000
2500
3000
Strike
Stage
α0 guess
α1 guess
Parabola
Reference

Calibration

Initial guess and calibrated smile for a May 2014 1m5y Swaption



$$u(T) = \int_0^T E^2(t,K)dt = \frac{e^{\rho\nu\alpha\Gamma(K)T} - 1}{\rho\nu\alpha\Gamma(K)}$$

The PDE becomes:

$$\frac{\partial V_{call}}{\partial u}(u,K) = \frac{1}{2}D^2(K)\frac{\partial^2 V_{call}}{\partial K^2}(u,K)$$

Hagan's local vol can be plugged into Andersen Ratcliffe expansion

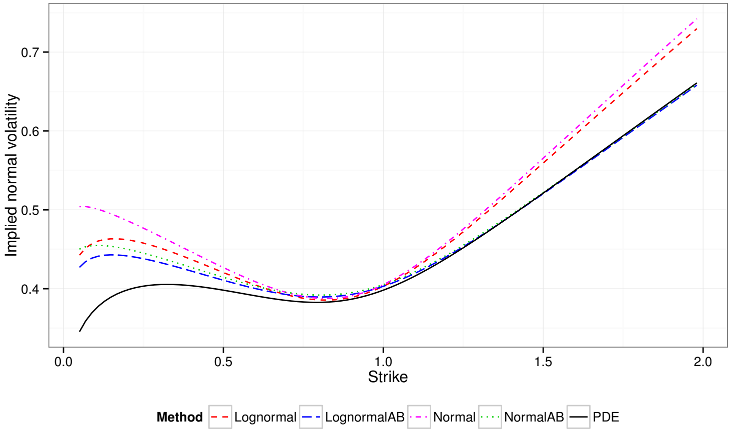
$$\Omega(u,K) = \Omega_0(K) + \Omega_1(K)u + \mathcal{O}(u^2)$$

with

$$\Omega_0(K) = \frac{\log\left(\frac{f+b}{K+b}\right)}{\int_K^f D^{-1}(k)dk}$$

$$\Omega_1(K) = -\frac{\Omega_0(K)}{\left(\int_K^f D^{-1}(k)dk\right)^2}\log\left(\Omega_0(K)\sqrt{\frac{(f+b)(K+b)}{D(f)D(K)}}\right)$$

Implied normal volatilities using parameters of Hagan (2013)
$\alpha = 0.35, \beta = 0.25, \rho = 0.25, \nu = 1, T = 2, f = 1$

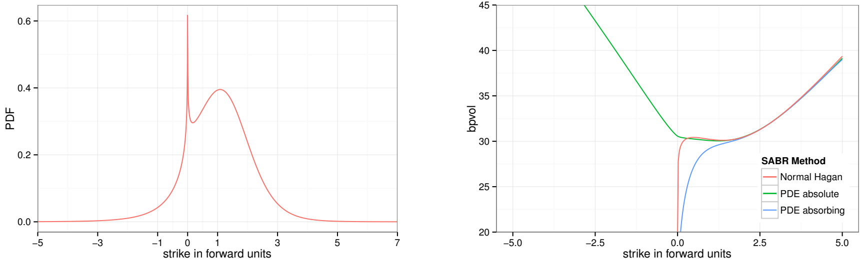


Free boundary SABR Antonov (2015):

$$C(F) = |F|^{\beta}$$

Easy to plug into Hagan's PDE:

- update C, Γ, y, D
- $F_{min} = -F_{max}$



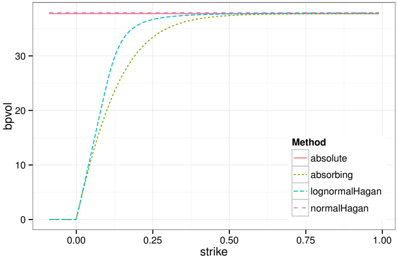
$f = 50$ b.p., $\beta = 0.1, \alpha = 0.5 f^{1-\beta}, \rho = -30\%, \nu = 30\%, \tau_{ex} = 3$

Free boundary SABR

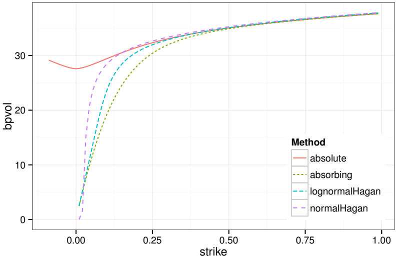
Absence of knee

ATM b.p. vol with $f = 1, \alpha = 0.35, \rho = 0\%, \nu = 100\%, \tau_{ex} = 1$

$\beta = 0$



$\beta = 0.1$



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