TR-BDF2 FOR STABLE AMERICAN OPTION PRICING

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Abstract. The Trapezoidal Rule with Second Order Backward Difference Formula (TR-BDF2) finite difference scheme is applied to the Black-Scholes-Merton PDE on a non uniform grid. American Option Convergence and Greeks stability is studied against studied against popular alternatives, namely Crank-Nicolson and Rannacher time-marching.

1. Introduction

It is well known that discontinuities in the payoff function or its derivatives can cause inaccuracies for numerical schemes when financial contracts are priced. For a vanilla (or a digital option), to avoid discretization errors, several ad-hoc remedies can be applied, for example, placing strike on node, or applying smoothing or projection techniques to the payoff [Pooley et al., 2003, Tavella and Randall, 2000]. Additionally, the scheme itself can introduce unwelcomed inaccuracies. The Crank-Nicolson scheme can introduce spurious oscillations in the greeks [Giles and Carter, 2006]. Rannacher time-marching is a known fix for european options. But we show here that it does not work as well for American (or Bermudan) options. In contrast, the Trapezoidal Rule with Second Order Backward Difference Formula (TR-BDF2) does not produce any spurious oscillations for European, Bermudan or American options, and is, like backward Euler (and unlike Crank-Nicolson), mathematically L-stable.

Estimating precisely the gamma and delta is key as those greeks are the most commonly used for hedging.

Where is TR-BDF2 being used? TR-BDF2 has been in use for more than 25 years in various domains. It was first used in electronics by Bank et al. to solve the coupled system of nonlinear partial differential equations that model the transient behavior of silicon VLSI device structures [Bank et al., 1985]. It remains a popular scheme in electronics [Gardner et al., 2004] and has been studied extensively [Malhotra Jogesh et al., 1994, Wild, 1993]. In biology, Tyson et al. used the scheme to solve a chemotaxis model [Tyson et al., 2000]. In mechanical engineering, Bathe studied its application for the transient response solution of structures when large deformations and long time durations are considered [Bathe, 2007].

The author is unaware of any use of TR-BDF2 within computational finance.

2. Background

The Black-Scholes-Merton partial differential equation is [Shreve, 2004]:

$$\frac{\partial f}{\partial t}(x,t) + \mu^*(x,t)x\frac{\partial f}{\partial x}(x,t) + \frac{1}{2}\sigma^*(x,t)^2x^2\frac{\partial^2 f}{\partial x^2}(x,t) = r(x,t)f(x,t) \tag{1}$$

Date: June 16, 2010.

Key words and phrases. Finite Difference, American Option, Crank-Nicolson, TR-BDF2.

where x is the underlying price, μ^* is the underlying drift, σ^* its volatility and r the interest rate. Using $\mu = \mu^* x$ and $\sigma = \sigma^* x$, this can be rewritten as:

$$\frac{\partial f}{\partial t}(x,t) + \mu(x,t)\frac{\partial f}{\partial x}(x,t) + \frac{1}{2}\sigma(x,t)^2\frac{\partial^2 f}{\partial x^2}(x,t) = r(x,t)f(x,t) \tag{2}$$

with terminal condition:

$$f(x,T) = F(x)$$

Let's discretize on the lattice defined by:

$$(x_i)_{i=0,..,m}\ ,\ \delta_i = x_i - x_{i-1}$$
$$(t_j)_{j=0,..,n}\ ,\ \zeta_j = t_j - t_{j-1}$$

The value of the function f in the lattice node (i,j) corresponding to the x-value x_i and the t-value t_j (i.e. $f(x_i,t_j)$) is denoted by $f_{i,j}$. Similarly, $\mu_{i,j}$ denotes $\mu(x_i,t_j)$, $\sigma_{i,j}^2$ denotes $\sigma(x_i,t_j)^2$, and $r_{i,j}$ denotes $r(x_i,t_j)$

Forward-looking difference operator:

$$\mathcal{D}_x^+ f_{i,j} = \frac{f_{i+1,j} - f_{i,j}}{\delta_{i+1}} \tag{3}$$

Backward-looking difference operator:

$$\mathcal{D}_x^- f_{i,j} = \frac{f_{i,j} - f_{i-1,j}}{\delta_i} \tag{4}$$

Central difference operator:

$$\mathcal{D}_x f_{i,j} = \frac{f_{i+1,j} - f_{i-1,j}}{\delta_{i+1} + \delta_i} \tag{5}$$

$$\mathcal{D}_x^2 f_{i,j} = 2\frac{\delta_i f_{i+1,j} - (\delta_{i+1} + \delta_i) f_{i,j} + \delta_{i+1} f_{i-1,j}}{\delta_i \delta_{i+1} (\delta_{i+1} + \delta_i)} \tag{6}$$

$$\mathcal{D}_t^- f_{i,j} = \frac{f_{i,j} - f_{i,j-1}}{\zeta_j} \tag{7}$$

$$\mathcal{D}_t^+ f_{i,j} = \frac{f_{i,j+1} - f_{i,j}}{\zeta_{j+1}} \tag{8}$$

3. TR-BDF2 Scheme

3.1. **What is TR-BDF2?** The Trapezoidal Rule with second order Backward Difference Formula (TR-BDF2) is a second order accurate fully implicit Runge Kutta method. It is a one step method that is L-stable. Crank-Nicolson itself is only A-stable [Dharmaraja, 2007]. In practice, the non L-stability manifests itself by spurious oscillations in the first and second space derivatives with Crank-Nicolson. This phenomenon will not appear with TR-BDF2 or with Backward Euler (which is also L-stable).

There are two stages. The first stage is the trapezoidal method (Crank-Nicolson) applied from t_n to $t_{n+\alpha} = t_n + \alpha\zeta_{n+1}$. The second stage is the 2 steps BDF method applied to the first stage output and the initial data.

Let $\mathcal{L}$ be the Black-Scholes-Merton operator defined by:

$$\mathcal{L}(f(x,t),x,t)) = -\mu(x,t)\frac{\partial f}{\partial x} - \frac{1}{2}\sigma(x,t)^2\frac{\partial^2 f}{\partial x^2} + r(x,t)f(x,t) \tag{9}$$

We have:

$$\frac{\partial f}{\partial t}(x,t) = \mathcal{L}(f(x,t),x,t) \tag{10}$$

The TR-BDF2 method can be written as [LeVeque, 2007]:

$$f^* = f_n + \frac{\alpha\zeta}{2}\left(\mathcal{L}(f_n) + \mathcal{L}(f^*)\right) \tag{11}$$

$$f_{n+1} = \frac{1}{2-\alpha}\left(\frac{1}{\alpha}f^* - \frac{(1-\alpha)^2}{\alpha}f_n + (1-\alpha)\zeta\mathcal{L}(f_{n+1})\right) \tag{12}$$

Even though there are 2 stages, this is still a 1 step method. Any full step only depends on the previous full step. This is an important difference with the standard second order backward difference scheme (BDF2) that depends on the two previous timesteps and can lose its accuracy [Windcliff et al., 2001] with variable timesteps and linear complimentary problems. This scheme does not suffer from such drawbacks.

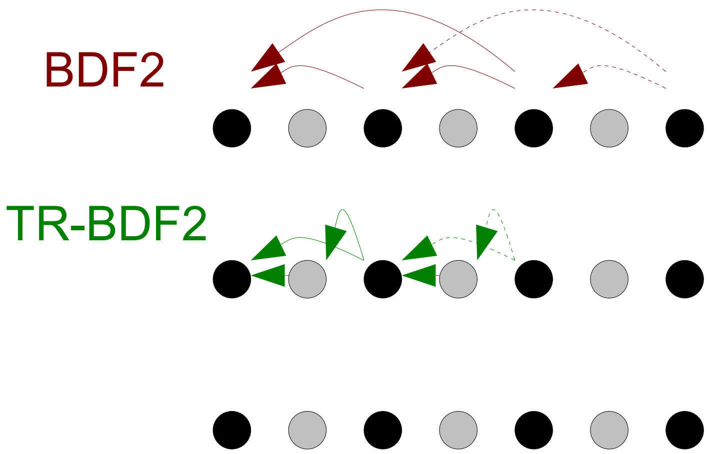


FIGURE 1. BDF2 has overlapping input over different time steps while TR-BDF2 has not

3.2. **Choice of α.**

There are 2 popular choices for α:

$$\alpha = \frac{1}{2}$$
$$\alpha = 2 - \sqrt{2}$$

The choice of $\alpha = \frac{1}{2}$ makes the equation simpler [LeVeque, 2007], while the choice of $\alpha = 2-\sqrt{2}$ is known to give the least truncation error among all α, proportional Jacobians [Bank et al., 1985], and the largest stability region [Dharmaraja et al., 2009]. With proportional Jacobians the underlying algorithm can be faster.

In practice, we did not find any significant difference in accuracy between the two when TR-BDF2 is applied to the Black-Scholes PDE.

3.3. **Discretization of the Black-Scholes PDE.** For $0 < i < m\ ,\ 0 \leq j < n$
Let $\mathcal{L}_{i,j} = (r_{i,j}I - \mu_{i,j}\mathcal{D}_x - \frac{1}{2}\sigma_{i,j}^2\mathcal{D}_x^2)$ be the discrete differential operator.
The Trapezoidal stage is:

$$\mathcal{D}_{\alpha t}^{+} f_{i,j+\alpha} = \frac{1}{2}(\mathcal{L}_{i,j+1}f_{i,j+1} + \mathcal{L}_{i,j+\alpha}f_{i,j+\alpha}) \tag{13}$$

where $\mathcal{D}_{\alpha t}^{+} f_{i,j} = \frac{f_{i,j+\alpha}-f_{i,j}}{t_{j+\alpha}-t_j} = \frac{f_{i,j+\alpha}-f_{i,j}}{\alpha\zeta_{j+1}}$

For $0 < i < m\ ,\ 0 \leq j < n$

$$a_{i,j+\alpha}f_{i-1,j+\alpha}+b_{i,j+\alpha}f_{i,j+\alpha}+c_{i,j+\alpha}f_{i+1,j+\alpha} = -a_{i,j+1}f_{i-1,j+1}+b_{i,j+1}^{*}f_{i,j+1}-c_{i,j+1}f_{i+1,j+1}$$

with:

$$a_{i,j} = \frac{1}{2\left(\delta_{i+1}+\delta_i\right)}\left(\mu_{i,j} - \frac{\sigma_{i,j}^2}{\delta_i}\right) \tag{14}$$

$$b_{i,j} = \frac{1}{\alpha\zeta_{j+1-\alpha}} + \frac{1}{2}\left(r_{i,j} + \frac{\sigma_{i,j}^2}{\delta_i\delta_{i+1}}\right) \tag{15}$$

$$c_{i,j} = -\frac{1}{2\left(\delta_{i+1}+\delta_i\right)}\left(\mu_{i,j} + \frac{\sigma_{i,j}^2}{\delta_{i+1}}\right) \tag{16}$$

$$b_{i,j}^{*} = \frac{1}{\alpha\zeta_j} - \frac{1}{2}\left(r_{i,j} + \frac{\sigma_{i,j}^2}{\delta_i\delta_{i+1}}\right) \tag{17}$$

The BDF2 stage is:

$$(2-\alpha)f_{i,j} - \frac{1}{\alpha}f_{i,j+\alpha} + \frac{(1-\alpha)^2}{\alpha}f_{i,j+1} = -(1-\alpha)\zeta_{j+1}\mathcal{L}_{i,j}f_{i,j} \tag{18}$$

$f_{i,j+\alpha}$ is the result of the first stage. For the second stage we have:

$$a_{i,j}f_{i-1,j} + b_{i,j}f_{i,j} + c_{i,j}f_{i+1,j} = \frac{1}{2-\alpha}\left(\frac{1}{\alpha}f_{i,j+\alpha} - \frac{(1-\alpha)^2}{\alpha}f_{i,j+1}\right)$$

with:

$$a_{i,j} = -\frac{1-\alpha}{2-\alpha}\zeta_{j+1}\frac{1}{\delta_{i+1}+\delta_i}\left(\frac{\sigma_{i,j}^2}{\delta_i} - \mu_{i,j}\right) \tag{19}$$

$$b_{i,j} = 1 + \frac{1-\alpha}{2-\alpha}\zeta_{j+1}\left(\frac{\sigma_{i,j}^2}{\delta_i\delta_{i+1}} + r_{i,j}\right) \tag{20}$$

$$c_{i,j} = -\frac{1-\alpha}{2-\alpha}\zeta_{j+1}\frac{1}{\delta_{i+1}+\delta_i}\left(\frac{\sigma_{i,j}^2}{\delta_{i+1}} + \mu_{i,j}\right) \tag{21}$$

4. Boundary Conditions

We consider here the boundary condition where we assume that $\frac{\partial^2 f}{\partial x^2} = 0$ at the boundaries. This is true for all payoffs linear at the boundaries, which is a reasonable assumption for most payoffs [Windcliff et al., 2004]. The Black-Scholes equation becomes:

$$\frac{\partial f}{\partial t}(x,t) + \mu(x,t)\frac{\partial f}{\partial x}(x,t) = r(x,t)f(x,t) \tag{22}$$

We will discretize the derivative by an order 1 in x approximation. This is reasonable because the first order error in x is proportional to the gamma, which we assumed to be 0:

$$\mathcal{D}_x^+ f_{i,j} = \frac{f_{i+1,j} - f_{i,j}}{\delta_{i+1}} \tag{23}$$

4.1. **Trapezoidal Stage.** Lower Boundary:

$$\mathcal{D}_{\alpha t}^+ f_{0,j+\alpha} = \frac{1}{2}(\mathcal{L}_{0,j+1} f_{i,j+1} + \mathcal{L}_{0,j+\alpha} f_{0,j+\alpha}) \tag{24}$$

$$b_{0,j+\alpha} f_{0,j+\alpha} + c_{0,j+\alpha} f_{1,j+\alpha} = b^*_{0,j+1} f_{0,j+1} - c_{0,j+1} f_{1,j+1}$$

with:

$$b_{0,j} = \frac{1}{\alpha\zeta_{j+1-\alpha}} + \frac{1}{2}\left(r_{0,j} + \frac{\mu_{0,j}}{\delta_1}\right) \tag{25}$$

$$b^*_{0,j} = \frac{1}{\alpha\zeta_j} - \frac{1}{2}\left(r_{0,j} + \frac{\mu_{0,j}}{\delta_1}\right) \tag{26}$$

$$c_{0,j} = -\frac{\mu_{0,j}}{2\delta_1} \tag{27}$$

Upper Boundary:

$$a_{m,j+\alpha} f_{m-1,j+\alpha} + b_{m,j+\alpha} f_{m,j+\alpha} = -a_{m,j+1} f_{m-,j+1} + b^*_{m,j+1} f_{m,j+1}$$

with:

$$a_{m,j} = \frac{\mu_{m,j}}{2\delta_m} \tag{28}$$

$$b_{m,j} = \frac{1}{\alpha\zeta_{j+1-\alpha}} + \frac{1}{2}\left(r_{m,j} - \frac{\mu_{m,j}}{\delta_m}\right) \tag{29}$$

$$b^*_{m,j} = \frac{1}{\alpha\zeta_j} - \frac{1}{2}\left(r_{m,j} - \frac{\mu_{m,j}}{\delta_m}\right) \tag{30}$$

4.2. **BDF2 Stage.** Lower boundary:

$$\mathcal{L}_{0,j} = (r_{0,j} I - \mu_{0,j}\mathcal{D}_x^+) \tag{31}$$

$$(2-\alpha)f_{0,j} - \frac{1}{\alpha} f_{0,j+\alpha} + \frac{(1-\alpha)^2}{\alpha} f_{0,j+1} = -(1-\alpha)\zeta_{j+1}\mathcal{L}_{0,j} f_{0,j}$$

We have:

$$b_{0,j} f_{0,j} + c_{0,j} f_{1,j} = \frac{1}{2-\alpha}\left(\frac{1}{\alpha} f_{0,j+\alpha} - \frac{(1-\alpha)^2}{\alpha} f_{0,j+1}\right)$$

with:

$$b_{0,j} = 1 + \frac{1-\alpha}{2-\alpha}\zeta_{j+1}\left(\frac{\mu_{0,j}}{\delta_1} + r_{0,j}\right) \tag{32}$$

$$c_{0,j} = -\frac{1-\alpha}{2-\alpha}\zeta_{j+1}\left(\frac{\mu_{0,j}}{\delta_1}\right) \tag{33}$$

Upper boundary:

$$a_{m,j}f_{m-1,j} + b_{m,j}f_{m,j} = \frac{1}{2-\alpha}\left(\frac{1}{\alpha}f_{m,j+\alpha} - \frac{(1-\alpha)^2}{\alpha}f_{m,j+1}\right)$$

with:

$$a_{m,j} = \frac{1-\alpha}{2-\alpha}\zeta_{j+1}\left(\frac{\mu_{m,j}}{\delta_m}\right) \tag{34}$$

$$b_{m,j} = 1 + \frac{1-\alpha}{2-\alpha}\zeta_{j+1}\left(-\frac{\mu_{m,j}}{\delta_m} + r_{m,j}\right) \tag{35}$$

5. American Option Specifics

The early exercise feature of the option adds a free boundary on top of the Black-Scholes partial differential equation. Let f be the option price, the following system of partial differential inequalities is satified [Lamberton and Lapeyre, 1996]:

$$\frac{\partial f}{\partial t}(x,t) + \mu(x,t)\frac{\partial f}{\partial x}(x,t) + \frac{1}{2}\sigma(x,t)^2\frac{\partial^2 f}{\partial x^2}(x,t) \leq r(x,t)f(x,t) \tag{36}$$

$$\left(\frac{\partial f}{\partial t}(x,t) + \mu(x,t)\frac{\partial f}{\partial x}(x,t) + \frac{1}{2}\sigma(x,t)^2\frac{\partial^2 f}{\partial x^2}(x,t) - r(x,t)f(x,t)\right)(f-F) = 0 \tag{37}$$

subject to:

$$f \geq F \tag{38}$$

$$f(x,T) = F(x) \tag{39}$$

There are many ways to solve the free boundary problem, the most popular being the Brennan and Schwartz algorithm [Brennan and Schwartz, 1977] (with known shortcomings [Jaillet et al., 1990]), Front-Tracking [Pantawopoulos et al., 1996], the Penalty Method [Nielsen et al., 2002], Operator Splitting [Ikonen and Toivanen, 2004], and Projected SOR [Wilmott et al., 1993].

The simplest way is to solve the tridiagonal system without considering the free boundary and to then apply the early exercise condition through `currentPrice = max(payoff, currentPrice)`. While this keeps the second order accuracy for explicit scheme (because the number of time steps has to be proportional to the square of the number of space steps for stability), it is only first order accurate in time in general [O'Sullivan and O'Sullivan, 2009].

As order-2 method, for the sake of simplicity, we will only consider Projected SOR. The results should be similar with other solving techniques. In the case of TR-BDF2, Projected SOR is applied at each stage.

6. Accuracy against Crank-Nicolson

6.1. **Convergence for a fixed space step.** As [O'Sullivan and O'Sullivan, 2009], we look at the convergence for a 1 year American Put option of strike K = 100 with a spot S = 100, a discount rate of 5%, and a volatility of 20%. We fix the space step size at 1.0. This corresponds to 500 space steps. Note that this places the strike and the spot on the grid, which is important to avoid introducing additional errors from the payoff in the grid [Pooley et al., 2003].

We use the same theoretical value of 6.0874933186 as in their paper. We verified its accuracy by an explicit scheme going progressively to 5 million timesteps (6.087493609042786). Note that this is not the exact american option price because we have fixed the space step size.

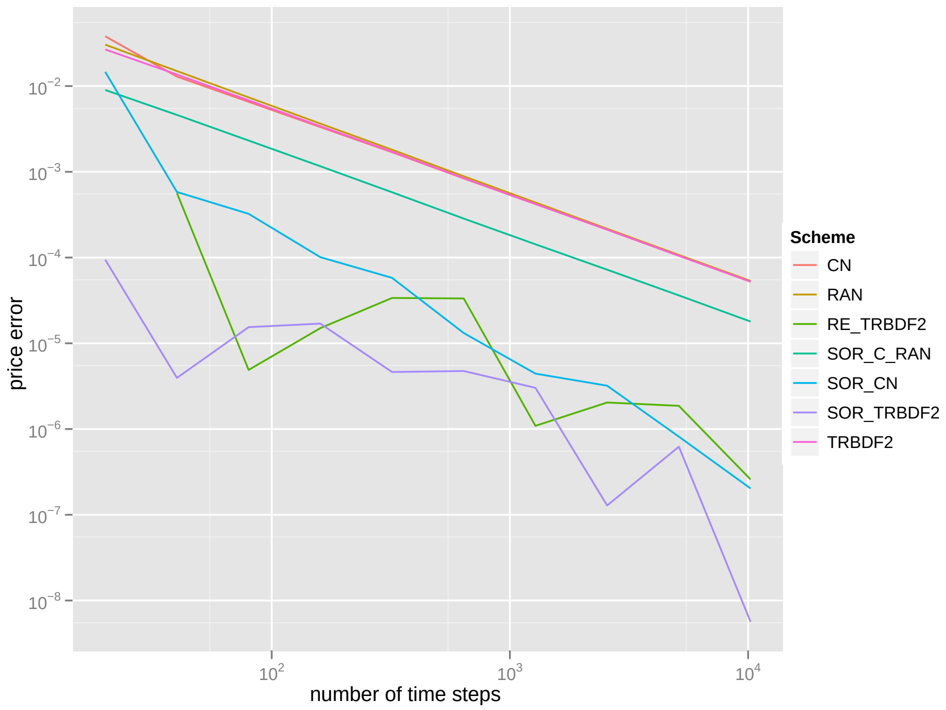


Figure 2. Convergence of TR-BDF2 and Crank-Nicolson for an American Put. CN denotes the Crank-Nicolson scheme with the order-1 approximation for the free boundary problem, TR-BDF2 is the TR-BDF2 scheme with the same order-1 approximation. SOR_CN is the Crank-Nicolson scheme solved by SOR, SOR_TRBDF2 is the TR-BDF2 scheme solved by SOR. RE_TRBDF2 is Richardson extrapolation in time applied to TR-BDF2 with the order-1 approximation. Finally, SOR_C_RAN is Rannacher time-marching procedure applied at each time step (see 7.2) solved by SOR.

Richardson extrapolation allows to gain an order in magnitude for the convergence of the order-1 free boundary approximation method. For the sake of simplicity we applied Richardson extrapolation the conventional way, not continuously like [O'Sullivan and O'Sullivan, 2009], but the conclusion would be the same with their method.

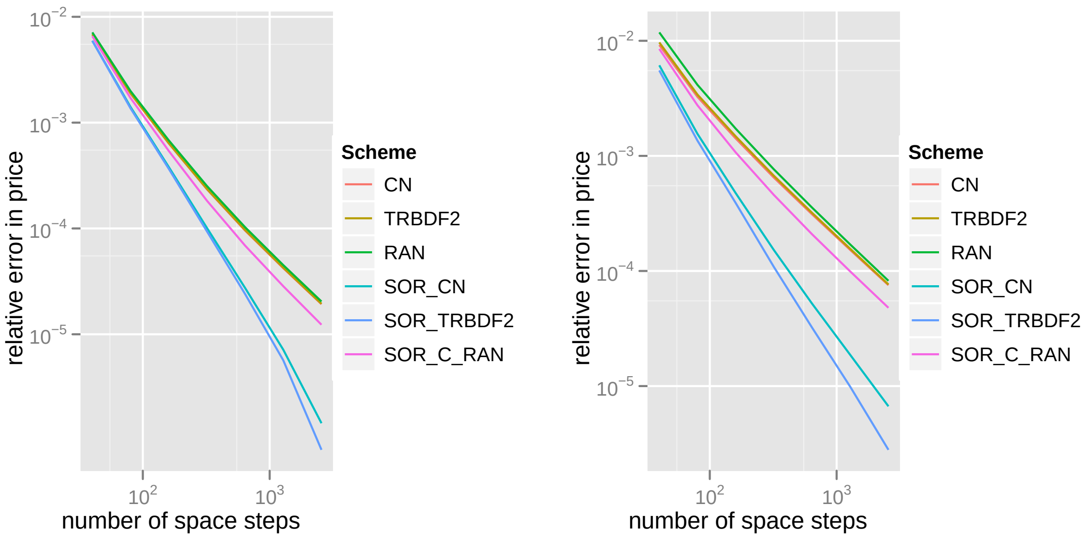
From Table 1, we see that TR-BDF2 is about twice as slow as Crank-Nicolson. This is because we have to solve twice per timestep. However, it is always as accurate or more accurate than Crank-Nicolson. We will see in the following sections why TR-BDF2 is an interesting scheme, even though it is slower.

6.2. **Convergence on Various Grid Geometries.** We now look at the convergence for a 1 year American Put option of strike $K = 100$ with a spot $S = 100$, a discount rate of 5%, and a volatility of 40%. We choose more practical boundaries for the grid:

$$x_m = Se^{3\sigma T}$$
$$x_0 = Se^{-3\sigma T}$$

It corresponds to 3 standard deviations up and down the spot. We also make sure that the strike is on the grid by slightly shifting the grid if necessary.We fix the ratio $\Lambda = \frac{m}{n}$ to 1 and 4 and compare the scheme maximum price error.

In our settings, $\Lambda = (x_m - x_0)\frac{dt}{dx}$. The reference is an american option price obtained by an explicit scheme on a very fine grid (4K space steps and 4M time steps). We pay attention to place the strike and the spot on the grid.



(A) $\Lambda = 1.0$

(B) $\Lambda = 4.0$

FIGURE 3. Maximum Error in Price of an American Put Option with a fixed Λ

Again, we see from Figure 8, that the convergence of TR-BDF2 is at least as good as the convergence of Crank-Nicolson. When Λ grows, TR-BDF2 maximum price error is lower because it does not oscillate like Crank-Nicolson. When Λ is small, the improvements of SOR vs the order-1 method is less apparent, because the number of timesteps grows enough to compensate the order-1 time precision.

6.3. **Accuracy on Random Grid Geometries.** We consider the same American Put Option as in 6.2 but we now select the the time and space discretisation randomly. This kind of test tells us how robust are the various schemes in a variety of practical situations. We compute the relative error in price of the various finite difference schemes on grids composed of 100 randomly chosen number of space steps and 100 randomly chosen number of time steps and look at the distribution of the error through a box-and-whisker plot (Figure 4).

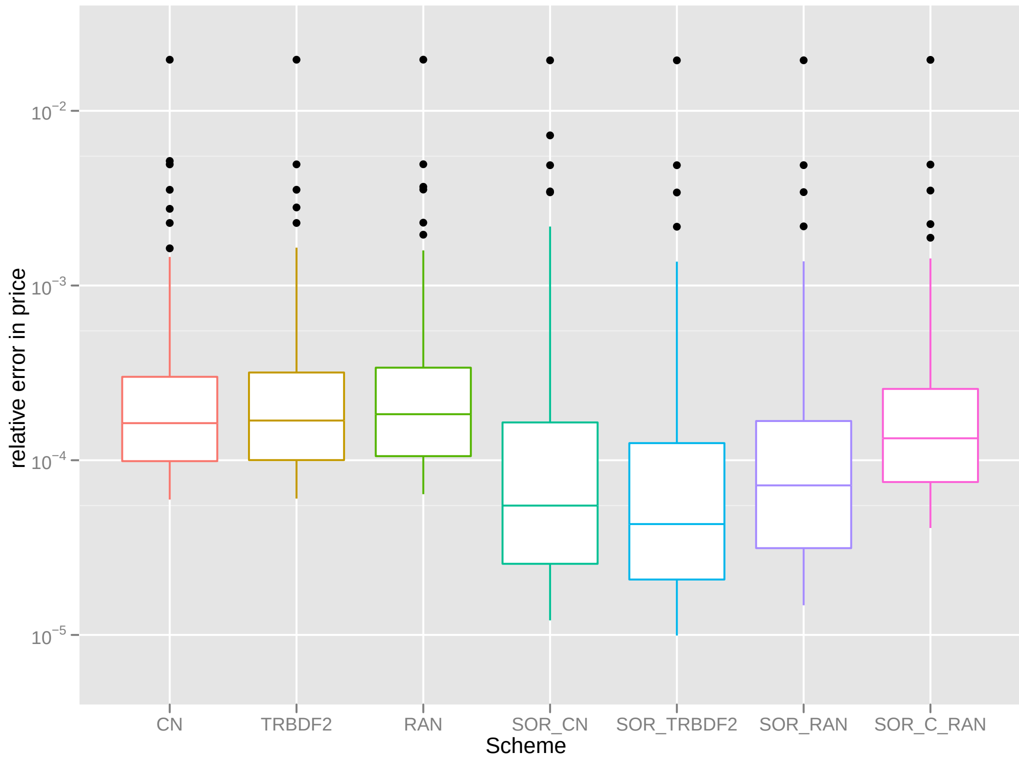


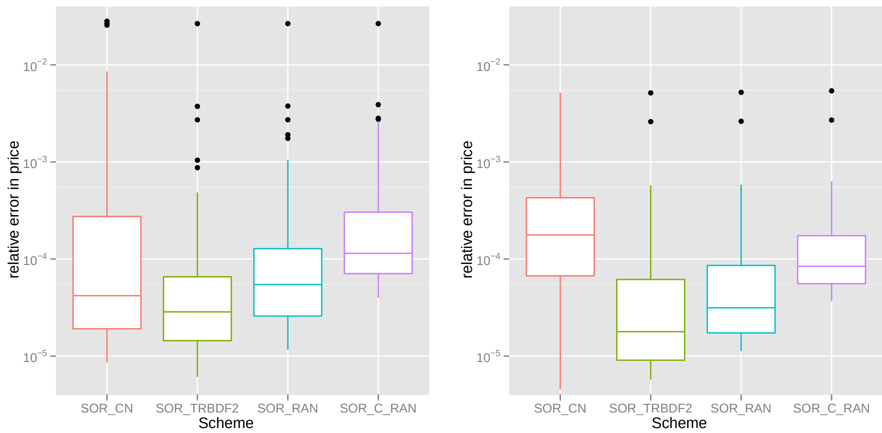
Figure 4. Error in the price of an American Put option on random grids

SOR_TRBDF2 distribution is lower and more compact than SOR_CN. SOR_RAN 75th percentile is lower than SOR_CN one because of the Rannacher time-marching improves Crank-Nicolson accuracy on grids with large Λ, but its 25th percentile is higher, because the few first backward euler steps are order-1 only and thus lessen the accuracy when Λ is low. SOR_C_RAN behaves similarly to SOR_RAN, albeit with a loss of the overall precision because the backward euler step is applied much more often. We know from 6.2 that the non-SOR solvers are only order-1 accurate, and don't allow to deduce much about the underlying schemes. Figure 4 confirms it.

6.4. **Accuracy on Random Non-Uniform Grid Geometries.** It is common for more complex payoffs to use a non uniform grid with more points near the payoff discontinuities [Tavella and Randall, 2000]. We consider the same test as in 6.3 but we now use a hyperbolic space discretization (with strike K on the grid):

$$
\begin{aligned}
x_i &= K + \alpha \sinh(c_1(1-\xi_i) + c_2\xi_i)\\
\xi_i &= \frac{i}{m}\\
c_1 &= \sinh^{-1}\frac{x_0 - K}{\alpha}\\
c_2 &= \sinh^{-1}\frac{x_m - K}{\alpha}
\end{aligned}
$$

where α is a parameter that determines the uniformity of the grid. The grid is highly non-uniform when $\alpha \ll x_m - x_0$, with a high concentration of points around the strike K. Figure 5a is similar to Figure 4. SOR_TRBDF2 error distribution



(A) $\alpha = 100$ (B) $\alpha = 5$

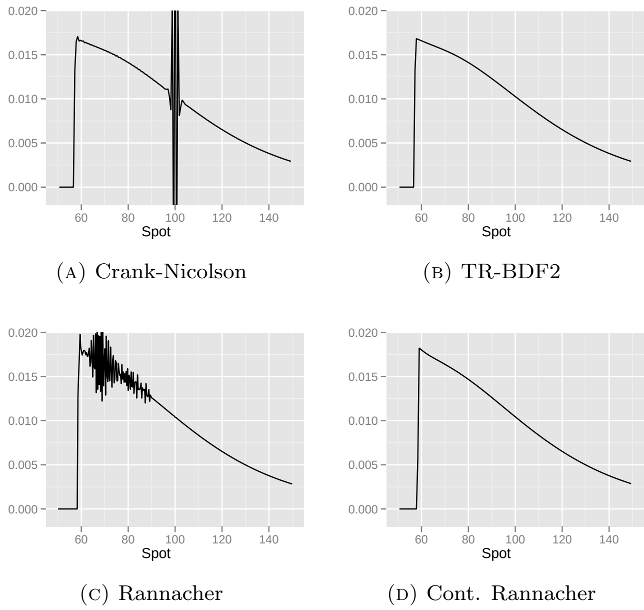
FIGURE 5. Error in the price of an American Put option on random non-uniform grids

is placed even lower than with a uniform grid and behaves better than SOR_CN or SOR_RAN. This is because with a non-uniform grid, the ratio $\frac{dx}{dt}$ can be much smaller than with a uniform grid when we consider the same number of space steps. In the case of SOR_CN, the price will therefore be perturbed by Crank-Nicolson oscillations more often. This is confirmed by Figure 5b where SOR_TRBDF2 clearly outperforms SOR_CN and SOR_RAN. Crank-Nicolson is not very well adapted to highly non-uniform grids and loses one order of accuracy in our settings.

7. Greeks Stability

7.1. **Gamma for an American Put.** It is well known that Crank-Nicolson can distort the greeks because of oscillations in the scheme [Giles and Carter, 2006]. TR-BDF2 does not have this issue. As an example, we give the Γ of an at the money 1 year American Put option of strike 100, 40% volatility , 5% interest rate on a grid composed of 500 space steps and 80 time steps. The boundary is set up the same way as in 6.2. The free boundary problem is solved by SOR.

The Γ with TR-BDF2 is smooth while Crank-Nicolson presents oscillations at the payoff discontinuity and Rannacher has oscillations near the early exercise. The Γ is smooth again when Rannacher time-marching is applied continuously. The At-the-money Rannacher graphs use 40 time steps while Crank-Nicolson uses

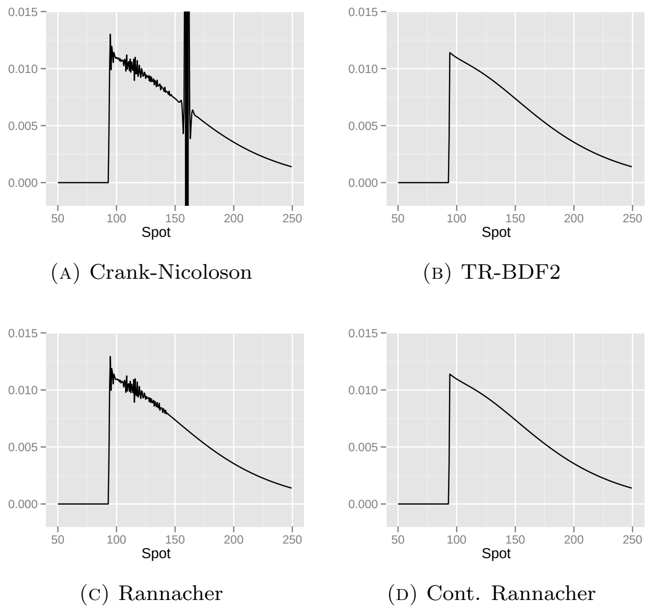


(A) Crank-Nicolson (B) TR-BDF2

(C) Rannacher (D) Cont. Rannacher

FIGURE 6. Γ of an At-The-Money American Put Option. Crank-Nicolson is shown with 80 time steps, others are with 40 time steps.

80 time steps, because the oscillations with Rannacher time-marching are smaller in amplitude and would be less visible with 80 time steps. TR-BDF2 is the same with 40 or 80 time steps.



(A) Crank-Nicoloson (B) TR-BDF2

(C) Rannacher (D) Cont. Rannacher

FIGURE 7. Γ of an In-The-Money American Put Option with strike at 160 and 80 time steps.

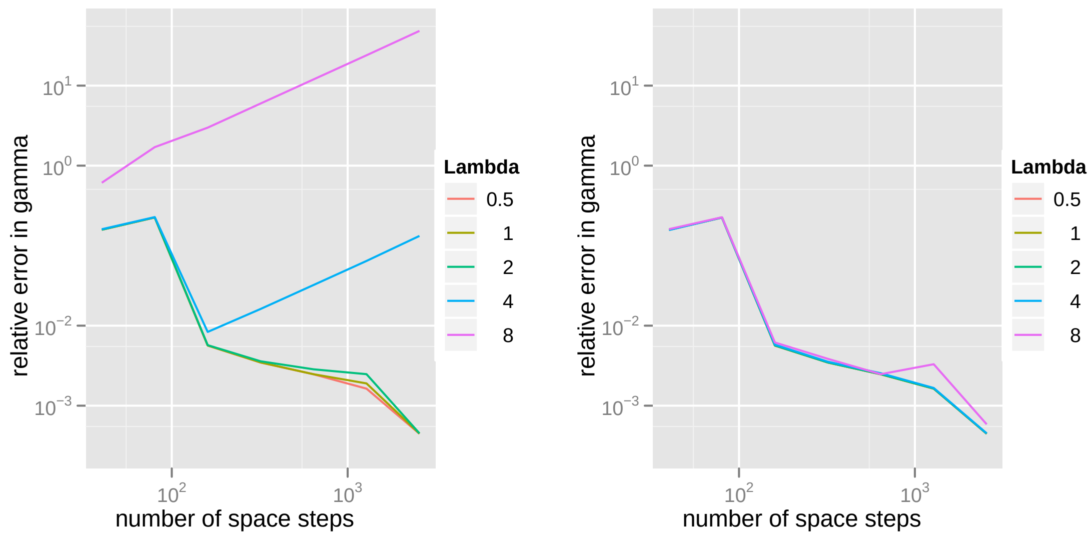
When the option is largely in the money, the early exercise oscillations are stronger, and visible with a lower Λ.

7.2. **The problem with Rannacher time-marching.** The Rannacher scheme introduces 2 or 4 half steps using backward Euler before doing Crank-Nicolson. This removes the oscillations in the greeks for european options [Giles and Carter, 2006]. But this is not true for American (or Bermudan) options as shown in Figures 6c and 7c.

Those oscillations were also present in the Crank-Nicolson scheme, but the oscillations resulting from the payoff discontinuity were much more important in comparison. The early exercise oscillations are stronger in Crank-Nicolson when the option is largely in the money. Figure 7a shows the phenomenon with a strike at 160 instead of 100 on 500 space steps and 80 time steps.

In order to have smooth greeks for American options, one would need to add backward Euler steps after every potential discontinuity introduced by the early exercise feature (see Figures 6d and 7d). The problem is that the resulting scheme would then be about 3 times slower as Crank-Nicolson (see Table 1) and lose 1 order of accuracy (see Figure 2).

7.3. **Gamma Accuracy on Various Grid Geometries.** For an option as simple as an American Put, the greeks will be acceptably smooth with Crank-Nicolson when the ratio $\Lambda = \frac{m}{n}$ is low enough. We measure the maximum relative error of Γ of the same option as in 6.2. We restrict the error computation to the spot interval [60, 300] because the discontinuity in the Γ before the spot reaches 60 would make the real maximum error not measurable (see Figure 8a). The reference Γ is obtained using an explicit scheme on a very fine grid. With TR-BDF2 the error



(A) Crank-Nicolson

(B) TR-BDF2

FIGURE 8. Maximum Relative Error of Γ for an American Put Option with various Λ

in Γ is independent of the ratio Λ while with Crank-Nicolson Γ is less precise as

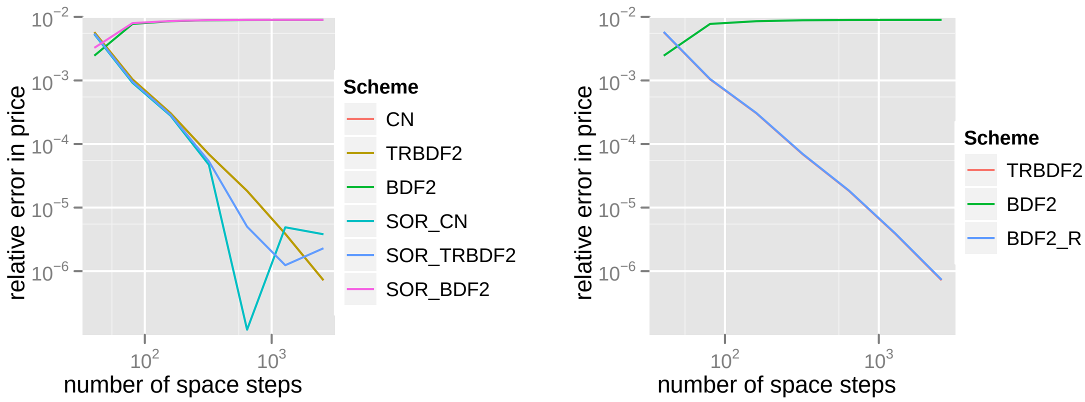
Λ grows. Interestingly with high Λ Crank-Nicolson loses some precision when the number of space steps increases.

8. Bermudan Option

8.1. **When BDF2 breaks.** The second backward difference (BDF2) scheme, another popular L-stable scheme, while simpler than TR-BDF2 has been shown not to be a good candidate in option pricing in general [Windcliff et al., 2001]. We give the example of a simple Bermudan option where BDF2 breaks down.

We consider an at the money 1 year Bermudan Put option of strike 100, 40% volatility , 5% interest rate than can be exercised at 6 months and at 1 year. The theoretical price of 13.386303 has been computed using an explicit finite difference scheme with more than 4 millions time steps and 5120 space steps. BDF2 is initialized with one backward Euler step.

The convergence is similar with SOR or with the order-1 free boundary approximation because there is only 1 date where the linear complimentary problem appears.



(A) TR-BDF2 vs Crank-Nicolson vs BDF2 (B) BDF2$_R$ vs BDF2

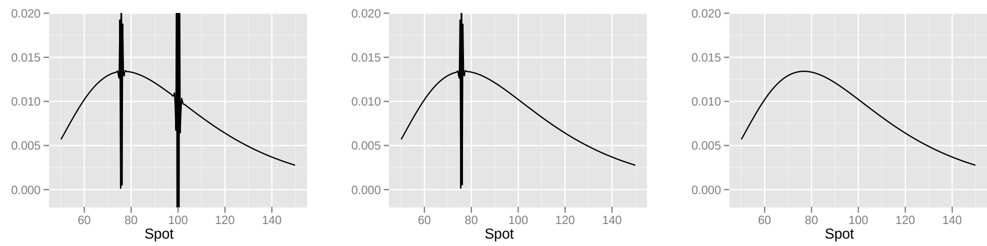
Figure 9. Relative Error of the price of a Bermudan Put Option on a grid with $\Lambda = 0.5$

BDF2 converges, but to a wrong price (13.506 instead of 13.386)! TR-BDF2 has no such issue and converges as well as Crank-Nicolson in an order-2 manner to the correct price. This example shows a key difference between BDF2, a multistep scheme, and TR-BDF2, a 1-step scheme.

In this simple example, there is a simple fix for BDF2, one need to restart BDF2 just before the early exercise date, i.e. to apply a backward Euler step instead of a BDF2 step at this date. We call this scheme BDF2$_R$. Another similar example where naive BDF2 fails is the case of an American option with discrete dividends, as the exercise boundary is then discontinuous. [Oosterlee et al., 2008] also fix this by restarting the scheme at each dividend date.

In more complex use cases like the Shout option in [Windcliff et al., 2001], the fix might not be so trivial. In contrast, TR-BDF2 just works.

8.2. **Gamma for a Bermudan Put.** We look again at Γ on a grid composed of 500 space steps and 80 time steps. We use the same at the money 1 year Bermudan



(A) Crank-Nicolson (B) Rannacher (C) TR-BDF2

FIGURE 10. Γ of an At-The-Money Bermudan Put Option.

Put option of strike 100, 40% volatility , 5% interest rate. The boundary is set up the same way as in 6.2.

Again, the Rannacher time-marching applied only at maturity does not fix all Crank-Nicolson oscillations. If we applied Rannacher time-marching at all bermudan exercise dates, we would then have a smooth Γ in a similar fashion to TR-BDF2. For Bermudan options, the Rannacher time-marching requires the knowledge of the early exercise dates to be efficient. [D'Halluin et al., 2001] observed the same phenomenon on Callable Bonds. In contrast, TR-BDF2 just works. This flexibility of TR-BDF2 can be important if one is led to solve other kind of linear complimentary problems.

9. CONCLUSION

We have shown how the TR-BDF2 scheme can be applied to option pricing. It does not suffer from Crank-Nicolson oscillations problem, particularly visible in the greeks. It is more resilient to the grid geometry and to the underlying PDE in general. And the Rannacher time-marching, while an interesting fix of Crank-Nicolson for European options, does not work as well for American options.

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TimeSteps	Scheme	Price	Error	Time(ms)
20	CN	6.04942861	3.81E-02	0.29
	TRBDF2	6.06077167	2.67E-02	0.53
	SOR_CN	6.07286091	1.46E-02	5.54
	SOR_C_RAN	6.07849059	9.00E-03	6.35
	SOR_TRBDF2	6.08758776	9.44E-05	6.18
40	CN	6.07460914	1.29E-02	0.53
	TRBDF2	6.07384972	1.36E-02	1.01
	SOR_CN	6.08691135	5.82E-04	5.92
	SOR_C_RAN	6.08288455	4.61E-03	7.47
	SOR_TRBDF2	6.08748937	3.95E-06	6.48
	RE_TRBDF2	6.08692777	5.66E-04	1.54
80	CN	6.08094463	6.55E-03	1.01
	TRBDF2	6.08067397	6.82E-03	2.07
	SOR_CN	6.08716972	3.24E-04	6.31
	SOR_C_RAN	6.08517794	2.32E-03	11.15
	SOR_TRBDF2	6.08747789	1.54E-05	9.03
	RE_TRBDF2	6.08749821	4.89E-06	3.07
160	CN	6.08417165	3.32E-03	2.10
	TRBDF2	6.08409115	3.40E-03	4.28
	SOR_CN	6.08739222	1.01E-04	9.56
	SOR_C_RAN	6.08633172	1.16E-03	18.68
	SOR_TRBDF2	6.08747636	1.70E-05	14.50
	RE_TRBDF2	6.08750834	1.50E-05	6.35
320	CN	6.08580185	1.69E-03	4.01
	TRBDF2	6.08577535	1.72E-03	8.23
	SOR_CN	6.08743534	5.80E-05	15.31
	SOR_C_RAN	6.08691611	5.77E-04	43.88
	SOR_TRBDF2	6.08749794	4.62E-06	25.98
	RE_TRBDF2	6.08745954	3.38E-05	12.52
640	CN	6.08665664	8.37E-04	7.92
	TRBDF2	6.08665101	8.42E-04	16.11
	SOR_CN	6.08748014	1.32E-05	27.65
	SOR_C_RAN	6.08720846	2.85E-04	66.32
	SOR_TRBDF2	6.08748857	4.75E-06	50.62
	RE_TRBDF2	6.08752666	3.33E-05	24.34
1280	CN	6.08707319	4.20E-04	16.27
	TRBDF2	6.08707162	4.22E-04	34.63
	SOR_CN	6.08748889	4.43E-06	53.06
	SOR_C_RAN	6.08735050	1.43E-04	125.21
	SOR_TRBDF2	6.08749030	3.02E-06	95.16
	RE_TRBDF2	6.08749223	1.09E-06	50.74
2560	CN	6.08728186	2.11E-04	32.16
	TRBDF2	6.08728145	2.12E-04	64.80
	SOR_CN	6.08749012	3.20E-06	98.03
	SOR_C_RAN	6.08742112	7.22E-05	266.09
	SOR_TRBDF2	6.08749319	1.28E-07	204.27
	RE_TRBDF2	6.08749128	2.04E-06	99.43
5120	CN	6.08738849	1.05E-04	66.09
	TRBDF2	6.08738832	1.05E-04	130.70
	SOR_CN	6.08749250	8.15E-07	222.25
	SOR_C_RAN	6.08745713	3.62E-05	519.11
	SOR_TRBDF2	6.08749394	6.22E-07	382.39
	RE_TRBDF2	6.08749518	1.86E-06	195.50
10240	CN	6.08744098	5.23E-05	133.65
	TRBDF2	6.08744095	5.24E-05	261.87
	SOR_CN	6.08749312	2.03E-07	400.15
	SOR_C_RAN	6.08747538	1.79E-05	1031.23
	SOR_TRBDF2	6.08749332	5.65E-09	757.89
	RE_TRBDF2	6.08749358	2.58E-07	392.57

TABLE 1. TR-BDF2 and Crank-Nicolson Convergence Table for an American Put, the reference value of 6.0874933186 has been obtained via [O'Sullivan and O'Sullivan, 2009]. CN denotes the Crank-Nicolson scheme with the order-1 approximation for the free boundary problem, TR-BDF2 is the TR-BDF2 scheme with the same order-1 approximation. SOR_CN is the Crank-Nicolson scheme solved by SOR, SOR_TRBDF2 is the TR-BDF2 scheme solved by SOR. RE_TRBDF2 is Richardson extrapolation in time applied to TR-BDF2 with the order-1 approximation. Finally, SOR_C_RAN is Rannacher time marching applied at each time step (see 7.2) solved by SOR.